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Insufficiency of the Ricci and Bianchi identities for characterising curvature

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Abstract. *The status of the Bianchi and Ricci identities as integrability conditions for the existence of a metric having a given tensor as Riemann curvature tensor is discussed. That these identities do not suffice to characterise curvature is shown by means of examples. Similar considerations apply to curvature forms in gauge theories.*

1. INTRODUCTION

The concept of curvature plays a very important role in physical theories such as general relativity and gauge theories. An obvious question which then arises is: which geometrical objects occur as curvatures in a given theory? For example one can ask which tensors K^{a}_{bcd} on a four-dimensional manifold are Riemann tensors of Lorentz metrics. In this case there are certain well-known «integrability conditions» which must be fulfilled, namely the Bianchi and Ricci identities. It seems to be widely believed that satisfaction of these conditions is also in some sense sufficient to ensure that a tensor is the Riemann tensor of some metric. In fact a closer examination of the problem reveals that it is not entirely clear in what sense these identities are integrability conditions. The problem is that the Bianchi and Ricci identities involve not only the curvature but also, throught the covariant derivatives which appear, the metric being sought. It seems that the only reasonable way to interpret such an identity as an integrability condition is to say that the integrability condition is satisfied if there exists *some* metric with respect to

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which the given tensor satisfies the identity.

If this interpretation of the Bianchi and Ricci identities is adopted then it turns out that these identities are not sufficient to characterise Riemann tensors, as is shown in what follows by means of examples. The relationship of these examples to the equivalence problem is then discussed. What has been said so far is open to the objection that there might exist an alternative way of interpreting the integrability conditions so that they would be sufficient to characterise curvatures. That this is highly unlikely is shown by another example. This concerns a tensor wich is not the Riemann tensor of any metric but which nevertheless has the property that given any point p there exists a metric defined in a neighbourhood of p whose Riemann tensor agress to infinite order with the given tensor at p.

The existence of examples such as those already mentioned has consequences for methods where one tries to find solutions of Einstein's equations with specific curvature properties by solving the Bianchi and Ricci identities. Such methods have been used by Ellis [3] and Held [6, 7]. They can be useful for obtaining solutions; the point being made here is that if one begins with a curvature and computes a metric using such a method it is still necessary to check explicitly that the metric obtained has the desired curvature. Of course it may be that in certain cases such as the case of the vacuum Einstein equations considered by Held such a check is not necessary. However this is something that requires proof. The examples of curvature candidates in gauge theories which satisfy the Bianchi identities but are not curvatures also suggest a problem concerning the attempt of Halpern [5] to formulate gauge theories using curvatures as the basic variables. His prescription for doing a path integral is to «integrate over the Bianchi identities» and it is not clear how this could be formulated so as not to include such false curvature candidates.

2. EXAMPLES

It is convenient to begin by defining a concise notation in which to express the Ricci identity. It will be written as $D^2 R = R * R$. Here D^2 denotes the skewed second covariant derivative, R denotes the Riemann tensor and $R*R$ is a quadratic expression in R. Now let g_{ab} be a metric (of any signature) of constant curvature on a manifold of dimension n greater than 2 and let R^a_{bcd} denote its Riemann tensor. Then $R^a_{bcd;e} = 0$. In particular $R^a_{bcd;[ef]} = 0$ and hence by the Ricci identity the expression $R * R$ vanishes. Let $\hat{R}^a_{bcd} = \alpha R^a_{bcd}$ for some constant α. $\hat{R}^a_{b[cd;e]} = \alpha R^a_{b[cd;e]} = 0$ and so $\hat{R}^a_{bcd}$ satisfies the Bianchi identity with respect to g_{ab}. It also satisfies the Ricci identity with respect to g_{ab} since $D^2 \hat{R} = \alpha D^2 R = 0 = \alpha^2 R * R = \hat{R} * \hat{R}$.

Suppose now that there exists a metric $\hat{g}_{ab}$ with curvature tensor $\hat{R}^a_{bcd}$. If we consider R^a_{bcd} and $\hat{R}^a_{bcd}$ as linear maps taking bivectors to (1,1)-tensors then both have rank $\frac{1}{2}n(n-1)$. It follows from results in [4] that $\hat{g}_{ab}$ is conformal to g_{ab}, say $\hat{g}_{ab} = e^{2U} g_{ab}$.

(In fact the result is only stated there for Lorentz metrics in 4 dimensions but it is easily generalised.) We now have:

$$\hat{R}_{ab} = \alpha R_{ab} = \frac{1}{4}\alpha R g_{ab} = (\frac{1}{4}\alpha R e^{-2U})\hat{g}_{ab}$$

This means that $\hat{g}_{ab}$ is an Einstein metric. Since the Ricci scalar of an Einstein space is constant we obtain $(\alpha R e^{-2U})_{,a} = 0$. Assuming $\alpha \neq 0, R \neq 0$ we then see that $U_{,a} = 0$. Hence e^{2U} is constant, the Riemann tensors are equal and $\alpha = 1$. It follows that for $\alpha \neq 1$ the tensor $\hat{R}^a_{bcd}$ is not the Riemann tensor of any metric.

Similar examples can be constructed in the context of gauge theories over a four-dimensional base manifold. It is known from the work of Mostow and Shnider [8] that in such theories a curvature F uniquely determines the connection it arises from in the generic case. Moreover if F is such a curvature and α is a non-zero constant then on algebraic grounds the only connection A which could possibly have αF as curvature is that giving rise to F. But if α is not equal to 1 then A obviously does not have curvature αF. Hence F in not the curvature of any connection. On the other hand it clearly satisfies the Bianchi identity with respect to A.

3. RELATION TO THE EQUIVALENCE PROBLEM

The equivalence problem is the problem of deciding whether two given spaces are isometric and this is generally done by examining curvature components. We now briefly review the formulation given in [2]. Given a metric on a manifold M let OM denote the corresponding bundle of orthonormal frames over M. There are two important geometrical objects defined on OM, namely the soldering form θ and the connection form ω. Isometries between two metrics are then in one to one correspondence with maps between their respective orthonormal frame bundles which map the connection and soldering forms into each other. The frame components of the Riemann tensor and its covariant derivatives define a collection of functions on OM. The Bianchi and Ricci identities can be expressed as polynomial relations between these functions. There are also relations between the functions, their exterior derivatives and the forms θ and ω which express the fact that certain things are covariant derivatives of certain others. The tensor $\hat{R}^a_{bcd}$ and its covariant derivatives with respect to g_{ab} define a collection of functions on the orthonormal frame bundle of g_{ab} which satisfy all the above mentioned relations and yet do not coincide with the functions arising from the Riemann tensor of g_{ab}. It seems that there is no way to tell that the collection of functions arising from $\hat{R}^a_{bcd}$ is not that arising from the Riemann tensor without computing this Riemann tensor directly.

In the general case once more, denote by $\mathcal{R}$ the set of functions defined on OM by the Riemann tensor and its covariant derivatives. Suppose for simplicity that the set

$\{df : f \in \mathcal{R}\}$ has the same rank r at each point of OM and that it is possible to choose a subset $\{f_1, \ldots, f_r\}$ of $\mathcal{R}$ so that the span of the derivatives df_i is r at each point. The f_i are said to form a maximal functionally independent set. Let $\mathcal{R}_j$ be the subset of $\mathcal{R}$ arising from covariant derivatives of order up to j. Then the f_j all belong to $\mathcal{R}_j$ for some j. It follows that all functions in $\mathcal{R}_{j+1}$ can be expressed as functions of the f_i in a neighbourhood of a given point of OM. This gives a map from $\mathbf{R}^r$ to $\mathbf{R}^q$ for some q which will be referred to as a shape. There is then a uniqueness theorem which says roughly the following (for a precise statement see [2]). Let f_i be a maximal functionally independent set for the metric g_{ab} as above and suppose that we have a map from M to another manifold such that the functions f_i' corresponding to the f_i under this map form a maximal functionally independent set of curvature functions for a metric g'_{ab}. Suppose further that the shapes of the two metrics are the same. Then g_{ab} and g'_{ab} are locally isometric. One could say that the maximal functionally independent set of curvature functions and the shape together determine the geometry. Consider once again the constant curvature metric. In that case the shape is trivial $(r = 0)$. The functions associated to the «non-curvature» $\hat{R}^a_{bcd}$ give rise to the same (trivial) shape and are not associated with any geometry. Of course this is a very special case and it is possible that under some assumption of genericity it is possible to prove existence of a geometry corresponding to a given shape. However no such theorem yet exists.

4. FURTHER EXAMPLES

Let h_{ab} be a smooth positive definite metric on $\mathbf{R}^2\backslash\{O\}$ whose conformal class cannot be extended continuously to $\mathbf{R}^2$. Let R^a_{bcd} be its Riemann tensor and suppose that this tensor is non-zero on an open dense subset of $\mathbf{R}^2\backslash\{O\}$. Let f be a function on $\mathbf{R}^2$ which is non-zero away from the origin but vanishes sufficiently fast near the origin so that $K^a_{bcd} = fR^a_{bcd}$ has a smooth extension to the whole plane and moreover all partial derivatives of the extension vanish at the origin. For instance we could take the components of the metric to be rational functions and f to be $e^{-1/(x^2+y^2)}$. As indicated in section 2 a Riemann tensor candidate of maximal rank uniquely determines the conformal class of any metric it could arise from. Hence if K^a_{bcd} were the Riemann tensor of any metric this metric would have to be conformal to h_{ab} on a dense subset of $\mathbf{R}^2$. By continuity it would then have to be conformal to h_{ab} on a dense subset of $\mathbf{R}^2\backslash\{O\}$. But this is incompatible with the fact that it has a continuous conformal class on $\mathbf{R}^2$. It follows that K^a_{bcd} is not the Riemann tensor of any metric on $\mathbf{R}^2$. In two dimensions the Bianchi identity is trivial because it involves antisymmetrisation over three indices. As to the Ricci identity, the expression earlier denoted by $K * K$ vanishes automatically in two dimensions. It follows that K^a_{bcd} satisfies both the Bianchi and Ricci identities with respect to the flat metric. In fact is not too difficult to show that if p is a point of the plane other than the origin then there exists a metric defined in a neighbourhood of p whose

Riemann tensor agrees with K^a_{bcd} on this neighbourhood. For a proof see DeTurck [1], p. 528. At the origin itself K^a_{bcd} agrees to infinite order with the Riemann tensor of the flat metric. Thus K^a_{bcd} is an example of a tensor which is not itself a Riemann tensor but which nevertheless agrees to infinite order with a Riemann tensor at each point.

From this two-dimensional example it is also possible to construct examples in higher dimensions. Let K^a_{bcd} be as before and let g_{ab} be a Lorentz metric on $\mathbb{R}^2$ whose Riemann tensor P^a_{bcd} never vanishes. Then the tensor $K^a_{bcd} + P^a_{bcd}$ on $\mathbb{R}^4$ satisfies the Ricci and Bianchi identities with respect to the product of the flat positive definite metric with g_{ab}. It also has the property of agreeing with a Riemann tensor to infinite order at each point. It remains to show that this tensor is not the Riemann tensor of any metric. Its rank is not maximal and so it is not possible to use the result previously mentioned on unique determination of conformal class. However a further result in [4] shows that any metric having this tensor as curvature must be of the form $\phi h_{ab} + \psi g_{ab}$ for some functions ϕ and ψ on a dense subset of $\mathbb{R}^4$. This is however inconsistent with the continuity of the metric as before and so no such metric exists.

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